A LOCAL STABILITY CRITERION FOR MONODISPERSE QUARTZ FRACTIONS IN THE COMPLETELY DRY STATE

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A statistical treatment is presented for the stability loss in monodisperse fractions of β-quartz from soils in the completely dry state. A stability criterion has been derived, which is compared with the Coulomb–Mohr criterion.

Most natural soils contain β-quartz in the dispersed state, which sometimes constitutes 90% or more of the total weight, and thus largely determines the physical properties.

This explains the theoretical interest in β-quartz fractions in the statistical mechanics of soils [1, 2]; an explicit analytical expression has been derived for the free energy F_0^i for a monodisperse β-quartz fraction consisting of spherical particles of an arbitrary degree i of dispersion:

$$F_i^0 = -\frac{3}{2}\,kTN_i^0\ln(2\pi m_i kT) + \frac{z_i}{2}\left[\frac{A_B^{(i)}N_i^{0^{\frac{7}{3}}}}{(\tilde{V}^{1/3}-N_i^{1/3}\delta_i)^4} - \frac{A_G^{(i)}N_i^{0^{\frac{4}{3}}}}{\tilde{V}^{1/3}-N_i^{0^{1/3}}\delta_i}\right] - kTN_i^0\ln\left(\varkappa\,\frac{\tilde{V}}{N_i^0}\right), \quad (1)$$

where T is absolute temperature; m_i, δ_i, and N_i^0 are the mass, diameter, and number of particles; k is Boltzmann's constant; z_i is the number of nearest neighbors of a particle; $\tilde{V}$ is the volume of a fraction; and κ_i is the structural parameter. Table 1 gives the interaction constants $A_B^{(i)}$ and $A_G^{(i)}$.

A particular case of this general equation has been considered [1], and it has been shown that one can neglect the first and last terms on the right in (1) with reasonable accuracy for fairly large particles up to T of about 300-400°K, and thus the equation can be extended to space conditions (T = 0 and pressure p = 0) and terrestrial ones (T ~ 300°K and p = 1 atm).

We draw up from (1) the necessary and sufficient conditions for thermodynamic equilibrium [3]:

$$\delta^2 F > 0 \quad \text{(subscripts omitted)}, \quad (2)$$

which enables us to analyze the stability of equilibrium states of the fundamental equation $p = -(\partial F/\partial V)_N$, which takes the following explicit form:

$$p = \frac{zN}{6V}\left(\frac{4A_B a}{h^5} - \frac{A_G a}{h^2}\right), \quad (3)$$

where a is the pitch and h is the shortest distance between the surfaces of particles [2].

In the case of a β-quartz fraction, the diagram corresponding to (3) is shown in Fig. 1, and stability analysis amounts to defining the roots of the polynomial f(h), which is of fourth degree (for brevity we omit here the explicit form of this polynomial). Apart from repeated roots, f(h) has only one real positive root of physical interest, which defines two regions of equilibrium states: the stability region $(\partial p/\partial h)_N < 0$ and the instability region $(\partial p/\partial h)_N > 0$.

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TABLE 1. Born and Gamaker Constants $A_B^{(i)}$ and $A_G^{(i)}$ for β-Quartz and Feldspars

Fraction No. i	Size range, cm	Mean diameter, δ_i, cm	$A_B^{(i)}=\frac{B\delta_i}{840}$, erg·cm^4	
			β-quartz	albite
1	10^{-7}—$5\cdot10^{-7}$	$2{,}5\cdot10^{-7}$	$0{,}75\cdot10^{-44}$	$1{,}1\cdot10^{-44}$
2	$5\cdot10^{-7}$—10^{-6}	$7{,}5\cdot10^{-7}$	$2{,}5\cdot10^{-44}$	$3{,}3\cdot10^{-44}$
3	10^{-6}—10^{-5}	$5\cdot10^{-6}$	$1{,}65\cdot10^{-43}$	$2{,}2\cdot10^{-43}$
4	10^{-5}—10^{-3}	$1\cdot10^{-4}$	$0{,}33\cdot10^{-41}$	$4{,}4\cdot10^{-42}$
5	10^{-3}—10^{-1}	$1\cdot10^{-2}$	$0{,}33\cdot10^{-39}$	$4{,}4\cdot10^{-40}$

Fraction No. i	$A_B^{(i)}=\frac{B\delta_i}{840}$, erg·cm^4	$A_G^{(i)}=\frac{A\delta_i}{24}$, erg·cm^4		
	orthoclase	β-quartz	albite	orthoclase
1	$2{,}3\cdot10^{-44}$	$0{,}56\cdot10^{-20}$	$0{,}52\cdot10^{-20}$	$1{,}0\cdot10^{-20}$
2	$7{,}1\cdot10^{-44}$	$1{,}6\cdot10^{-20}$	$1{,}5\cdot10^{-20}$	$3{,}0\cdot10^{-20}$
3	$4{,}7\cdot10^{-43}$	$1{,}1\cdot10^{-19}$	$1{,}0\cdot10^{-19}$	$2{,}0\cdot10^{-19}$
4	$9{,}4\cdot10^{-42}$	$0{,}22\cdot10^{-17}$	$0{,}21\cdot10^{-17}$	$0{,}4\cdot10^{-17}$
5	$9{,}4\cdot10^{-40}$	$0{,}22\cdot10^{-15}$	$0{,}2\cdot10^{-15}$	$0{,}42\cdot10^{-15}$

The diagram shows clearly that the states of dried β-quartz fractions are substantially dependent on δ and the structure, since (3) contains the structural parameters z, a, and h, which are rather microstructural parameters in the sense of the definitions used in soil science. The diagram shows that the stability limit is represented by point s, the root of f(h) that lies in the region of negative pressures. This explains effects such as the loss of carrying capacity in sandy soils, the vibration effect, and the fluidization of sands [4].

One naturally supposes that a mixture (for instance, an artificial soil mixture, a powder, etc.) of macroscopic and microscopic fractions would have a transitional region of states, which will be seen on the diagram as a set of points similar to point s.

A very important point for applications is theoretical explanation of the empirical Coulomb–Mohr soil strength criterion [5]:

$$\tau \leqslant C + f\sigma, \qquad (4)$$

since it is based on a simple hypothesis that Amonton's (1699) empirical law for dry friction, which applies for bodies of various compositions and natures, can be extended to powders and bound soils [Coulomb (1773)]. Afterwards, Mohr (1933) gave an illustration of (4) from the viewpoint of the theory of elasticity by means of the envelope of a family of circles (the Mohr circles), but this also did not serve to verify the hypothesis.

Statistical mechanics provides a fundamental physical theory of the structure and properties of matter, and thus provides a certain advance in this direction; the state diagram given by statistical methods gives a completely natural identification of the quantity C and the pressure corresponding to the point s in the state diagram [in the literature, (4) is sometimes put in the form $\tau \leq f(C' + \sigma)$ [5], where $C = fC'$].

This physical interpretation of C corresponds to the binding pressure concept (for adhesion, or attachment) [6] for any type of soil; the diagram indicates that C is the limiting negative pressure that the β-quartz fraction will withstand without loss of stability. The physical meaning of C is thus governed solely by the nature of the interaction forces (here between the β-quartz particles).

The diagram also gives rise to an alternative assertion: either $p > 0$ (stable equilibrium) or $p < 0$, and then the fraction can pass over to the region of unstable states via the value $K = 0$ (K is the bulk modulus [1]) or $p = |p(h = s)|$, which is the same.

As the thermodynamic condition $\delta^2 F > 0$ is not only sufficient, but also necessary, for stable equilibria (and in accordance with the theory of elasticity, the pressure is equal to 1/3 of the first invariant of the stress tensor), we can give the following formulation for the local stability criterion for dried monodisperse β-quartz soil fractions.

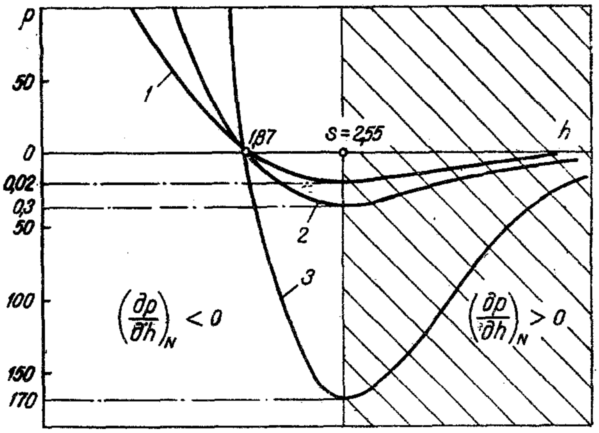

Fig. 1. State diagram for β-quartz fractions of sizes δ (m): 1) 10^{-4}; 2) 10^{-5}; 3) $5 \cdot 10^{-8}$; p, N/m^2; h, 10^{-10} m; N, number of particles.

The necessary and sufficient condition for a fraction to pass from the region of stable equilibrium states to that of unstable equilibrium states is that the first invariant I_1 of the stress tensor reach the negative-pressure point, which is three times the absolute value of the extremal pressure p(h = s) on the mechanical-state diagram, i.e.,

$$I_1 = 3p\,(h = s). \tag{5}$$

Some practical consequences follow from the formulation of the criterion. First of all, in choosing the permissible pressure p(h = s) we need to analyze the equation for the mechanical state. Such equations are generally lacking for a large range of chemical and mineral compositions of individual soil fractions, and also for soils as a whole, so it is necessary to derive them.

The most important practical consequence of the criterion is the following. The form of Eq. (5) indicates that one can determine the stability of a β-quartz fraction at a point by knowing only the first invariant of the stress tensor at that point, which is a simple number. This results in a mathematical simplification in the formulation of the boundary conditions for the limiting equilibrium in the case of β-quartz, and hence simplifies solution of the problem as a whole.

In conclusion, we consider the two criteria and enumerate some disadvantages of the Coulomb–Mohr criterion.

From (5) and simple transformation of (4) we can show for β-quartz sand why there is no practical value in the Coulomb–Mohr criterion; in fact, (4) for the limiting state can be put as

$$\tau - f\sigma = C.$$

As we have put C ≡ p(h = s) above, we have in accordance with (5) that the left side of this expression can be equated to the first invariant I_1 of the stress tensor apart from the factor 1/3, i.e.,

$$\tau - f\sigma = \frac{1}{3} I_1. \tag{6}$$

This means that to determine the stability at a given point for β-quartz sand there is no need to know the coefficient f of the dry friction for β-quartz, and this is all the more desirable, since at present there is no proper theoretical explanation of the nature of friction [6]. It is quite sufficient that we know a quantity characterizing the dependence of the limiting state at a point on the chemical and mineral composition, structure and nature of the particles, and so on (see above for the fundamental equation for the mechanical state of a pure β-quartz fraction; also see [4]). Then (6) allows us to consider τ − fσ as the fundamental concept of the first invariant of the stress tensor.

Remark. Attempts of intuitive type have been made in soil mechanics to avoid using τ − fσ; it has been considered [7] that many unsuccessful formulations in soil mechanics would vanish if one abandoned the attempt to divide the overall resistance to shear in a clay soil into components that do not really exist in nature, namely, adhesion and friction.

From the viewpoint of the local stability criterion, in the case of β-quartz the question is not whether we should divide up the overall shear resistance, but whether we should replace $\tau - f\sigma$ by a single quantity in accordance with (6), which is independent of the degrees of dispersion, and thus applies to a highly dispersed material (clay) or a coarse one (sand).

Finally, there is a difference in the character of the two criteria. While (4) is purely empirical, the question of whether it is necessary and sufficient has no relation to the formulation. On the other hand, the local stability criterion of (5) has a theoretical basis and a strict proof of its necessity and sufficiency (a consequence of the condition $\delta^2F > 0$).

LITERATURE CITED

1. Yu. S. Yudin, in: Transactions of the Siberian Branch of the V. I. Lenin All-Union Academy of Agricultural Sciences, SibIMÉ [in Russian], Vol. 1, No. 7, Pt. 2, Novosibirsk (1970), p. 80.
2. Yu. S. Yudin, in: Transactions of the Siberian Branch of the V. I. Lenin All-Union Academy of Agricultural Sciences, SibIMÉ [in Russian], Vol. 1, No. 7, Pt. 2, Novosibirsk (1970), p. 75.
3. V. K. Semenchenko, Selected Chapters in Theoretical Physics [in Russian], Moscow (1966).
4. Yu. S. Yudin, in: Transactions of the Siberian Branch of the V. I. Lenin All-Union Academy of Agricultural Sciences, SibIMÉ [in Russian], No. 8, Pt. 3, Novosibirsk (1972), p. 55.
5. N. A. Tsytovich, Mechanics of Soils [in Russian], Moscow (1963).
6. B. V. Deryagin, What Is Friction? [in Russian], Moscow (1963).
7. V. A. Florin, Principles of Soil Mechanics [in Russian], Moscow–Leningrad (1959).